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Realizability and the Axiom of Choice

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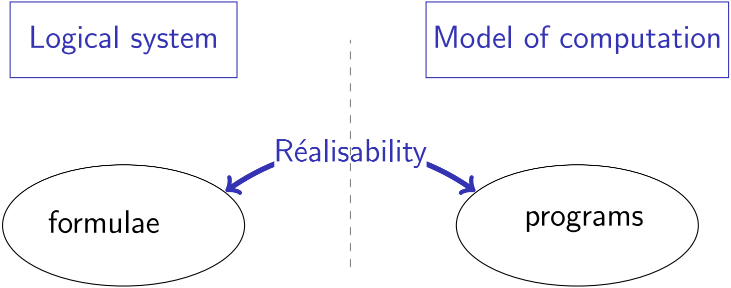
Réalisability

Goal:
To interpret the formulae of a logical system in a model of computation in order to extract the computational meaning of mathematical proofs.

Exemple: the tautology $A \Rightarrow A$ is realized by the programs which take an entry of type A and return a result of type A

Kleene 1945
Interprétation of Heyting arithmetic by sets of (indexes) of recursive functions.

From Curry Howard to Forcing



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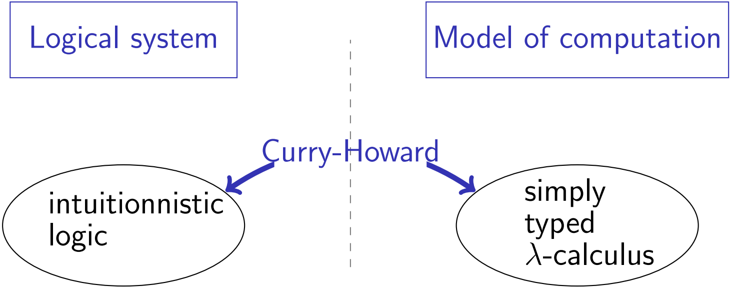

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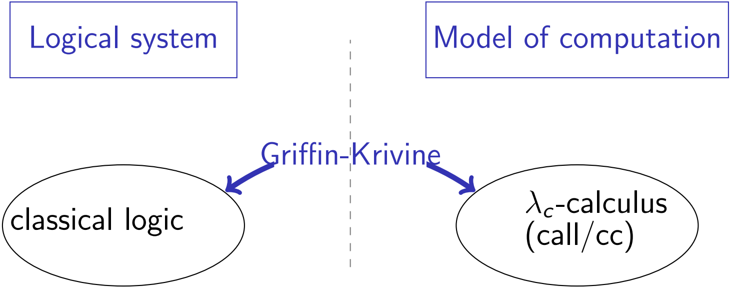

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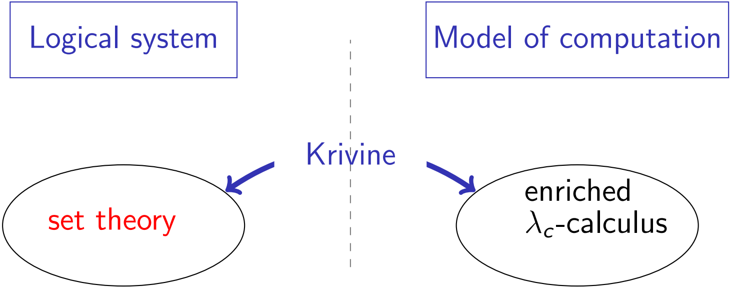

Forcing in short

Cohen 1963: Independence of the continuum hypothesis from set theory *ZF*

Forcing: technique for constructing models of ZFC and prove independence/consistency results

We consider a Boolean algebra $(\mathbb{B}, 0, 1, \leq, \wedge\vee, \neg)$, and we "evaluate" each formula of set theory by an element of $\mathbb{B}$.

$$p \Vdash \varphi$$

Basically, it means assigning a "degree of truth" to each formula φ

$$\text{Théorie} := \{\varphi \ : \ 1 \Vdash \varphi\}$$

forms a coherent classical theory which contains ZF(C) (if we do the process starting from a model of ZF(C))

Lambda calculus in short

λ-**terms**	t, u	$::=$	x	(variable)
		$\mid$	tu	(application)
		$\mid$	$\lambda x.t$	(abstraction, where x is a variable and t a λ-term)

Basically, $\lambda x.t$ means that we consider t as a function on x.
Exemple: $\lambda x.x$ is the identity

The β-reduction: $(\lambda x.t)u \rightarrow_\beta t[u/x]$

Exemple : $(\lambda x.x)t \rightarrow_\beta t$
(Informal) exemple: $(\lambda x.x + 1)3 \rightarrow_\beta 3 + 1$

Classical realizability in short

We use programs together with their environments (stacks) like forcing conditions

The idea

We assign to each formula φ

- a "degree of truth" $|\varphi|$ which is a set of programs (λ_c-terms)
- a "degree of falsity" $||\varphi||$ which is a set of stacks

These are defined simultaneously: $\xi \in |\varphi|$ if ξ is "incompatible" with every stack in $||\varphi||$ (and $||\varphi||$ is defined by induction on the lenght of the formula)

We write $\xi \Vdash \varphi$ for $\xi \in |\varphi|$

Realizability algebras

Instead of Boolean algebras, we use...

A realizability algebra

- Λ a set of programs (λ_c-terms)
- Π a set of stacks
- $\mathcal{R}$ a set of realizers ("trustful programs")
- $\prec_K$ the execution, a pre-order on processes $t * \pi$ (where t is a program and π is a stack)
- $\perp\!\!\!\perp$ the pole a $\prec_K$-final segment of the set of processes (it defines the "incompatible processes")

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$$\{\varphi |\ \exists \theta \in \mathcal{R}(\theta \Vdash |\varphi|)\}$$

is a classical coherent theory which contains ZF
(if we do the process starting from a model of ZFC)

Realizability algebras vs Forcing

Boolean algebras can be seen as special cases of realizability algebras.

- $\Lambda = \Pi = \mathbb{B}$
- $pq = p * q = p \wedge q$
- R=$\{1\}$
- $p \succ_K q \iff p \leq q$ (i.e. $p \wedge q = p$)
- $\perp\!\!\!\perp = \{0\}$

Non extensional set theory

In order to realize the axioms of set theory, we work with a *non-extensional* version of ZF, called ZF_ε.

We consider, two membership relations:

- $\in$ the usual extensional one
- ε a (strict) non-extensional one

... and two equality relations

- $\simeq$ the usual extensional one $x \simeq y \iff \forall z(z \in x \iff z \in y)$
- $=$ Leibniz identity, i.e. two sets are identical if the satisfy the same formulae

Non extensional set theory ZF_ε

0. **Extensionnality axiom:**

$$\forall x \forall y [x \in y \iff \exists z \varepsilon y \; (x \simeq z)];$$
$$\forall x \forall y [x \subseteq y \iff \forall z \varepsilon x \; (z \in y)].$$

1. **Axiom of pairing:** $\forall a \; \forall b \; \exists x \; (a \; \varepsilon \; x \wedge b \; \varepsilon \; x)$.
2. **Axiom of union:** $\forall a \; \exists b \; \forall x \varepsilon a \; \forall y \varepsilon x \; (y \; \varepsilon \; b)$.
3. **Axiom of power set:** for every formula $F(x, z_1, \ldots, z_n))$

$$\forall a \; \exists b \; \forall z_1 \ldots \forall z_n \; \exists y \varepsilon b \; \forall x \; (x \; \varepsilon \; y \iff (x \; \varepsilon \; a \wedge F(x, z_1, \ldots, z_n)))$$

4. **Replacement axiom:** for every formula $F(x, y, z_1, \ldots, z_n)$,

$$\forall z_1 \ldots \forall z_n \; \forall a \; \exists b \; \forall x \varepsilon a \; (\exists y \; F(x, y, z_1, \ldots, z_n) \Rightarrow \exists y \varepsilon b \; F(x, y, z_1, \ldots, z_n))$$

5. **Axiom of foundation:** for every formula $F(x, z_1, \ldots, z_n)$,

$$\forall z_1 \ldots \forall z_n \; \forall a \; (\forall x \; (\forall y \; \varepsilon \; x \; (F(y, z_1, \ldots, z_n) \Rightarrow F(x, z_1, \ldots, z_n))) \Rightarrow F(a, z_1, \ldots, z_n))$$

6. **Axiom of infinity:** for every formula $F(x, y, z_1, \ldots, z_n)$,

$$\forall z_1 \ldots \forall z_n \; \forall a \; \exists b \; (a \; \varepsilon \; b \wedge \forall x \varepsilon b \; (\exists y \; F(x, y, z_1, \ldots, z_n) \Rightarrow \exists y \varepsilon b \; F(x, y, z_1, \ldots, z_n)))$$

Ground model and the langage of realizability

We start with a model $\mathcal{M}$ of ZFC, the ground model, the realisability algebra lives in this model

The language of realizability

It's an extension of the language of ZF_ε where we add:

- a new constant symbol for each set of $\mathcal{M}$
- a new function symbol for every class function definable with parameters in $\mathcal{M}$

The truth and falsity values

For each formula φ of the languange of realizability we define by induction its truth value denoted $|\varphi|$ and its falsity value denoted $||\varphi||$.

- $|\varphi| = \{t \in \Lambda;\ \forall \pi \in \|\varphi\|\ (t * \pi \in \perp\!\!\!\perp)\}$
- $\|\top\| = \emptyset,\ \|\bot\| = \Pi,$
- $\|a \not\in b\| = \{\pi \in \Pi;\ (a, \pi) \in b\}$
- $\|a \subseteq b\|$ et $\|a \notin b\|$ sont définies simultanément:
  - $\|a \subseteq b\| = \{t \bullet \pi;\ (t, \pi) \in \Lambda \times \Pi,\ (c, \pi) \in a \text{ and } t \in |c \notin b|\}$
  - $\|a \notin b\| = \{t \bullet t' \bullet \pi;\ (t, t', \pi) \in \Lambda \times \Lambda \times \Pi,\ (c, \pi) \in b,\ t \in |a \subseteq c|,\ t' \in |c \subseteq a|\},$
- $\|A \Rightarrow B\| = \{t \bullet \pi; (t, \pi) \in \Lambda \times \Pi,\ t \in |A|, \pi \in \|B\|\},$
- $\|\forall x A\| = \{\pi \in \Pi;\ \exists a(\pi \in \|A[a/x]\|)\},$

We write $t \Vdash \varphi$ for $t \in |\varphi|$.

The realizability model

ZF_ε is a conservative extension of ZF.

Theorem

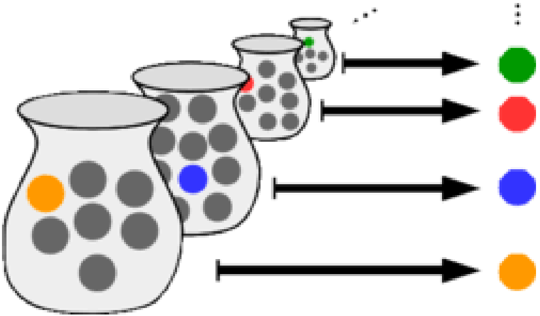
The set of formulas which are realized is a classical coherent theory which contains the axioms of ZF_ε (provided we assume the consistency of ZFC)

Exemple: the identity realizes the axiom of pairing; the axiom of foundation is realized by Turing fixed point combinator; ...

We call realizability model any model of such a theory (analogous to Boolean valued model). It yields a (actually many) structure in the language of ZF_ε denoted $\mathcal{N}_\varepsilon$ or $\mathcal{N}$, and a (actually many) structure in the language of ZF, denoted $\mathcal{N}_\in$.

The Axiom of Choice

Can we realize the axiom of choice?

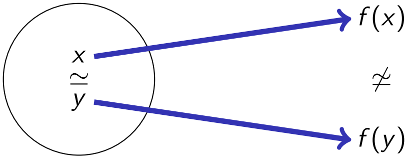


Non extensional axiom of choice

We can easily realize a non extensional version of AC, called NEAC
(by many different programs, for instance by the instruction 'quote')

NEAC: existence of a *non-extensional* choice function, i.e.

$$\begin{array}{lcl} x = y & \Rightarrow & f(x) = f(y), \text{ but} \\ x \simeq y & \not\Rightarrow & f(x) \simeq f(y) \end{array}$$



Réaliser l'axiome du choix

Krivine 2004

By using NEAC, we can realize DC

$$AC \iff \forall \kappa \in Ord \ (ZL_\kappa)$$
$$DC = ZL_\omega$$

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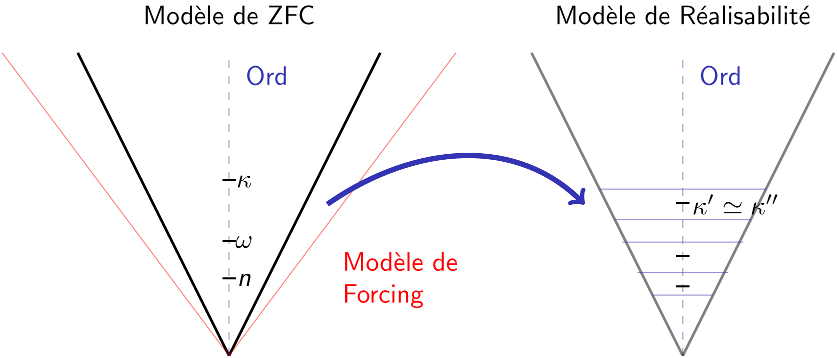
For every cardinal κ in a model of ZFC, we can construct a realizability model of $ZF + ZL_\kappa$

Zorn's lemma restricted to an ordinal

Zorn's lemma restricted ZL_κ

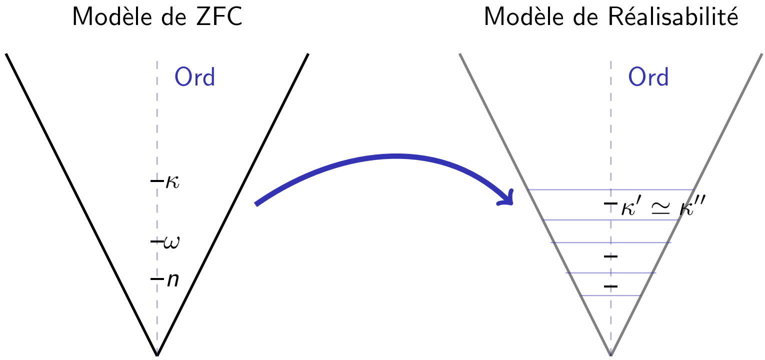
Let X be a non-empty set, and let R be a binary relation on X such that for every $\alpha < \kappa$, every R-chain $s = (s_\beta)_{\beta<\alpha}$ of length α can be extended (i.e. one can find an element $y \in X$ such that $s_\beta \; R \; y$ for every $\beta < \alpha$), then there is an R-chain of length κ.

What is the problem?



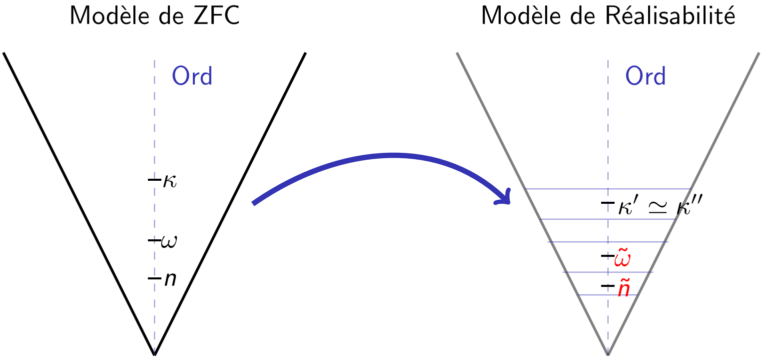
Unlike Forcing models, realizability models are not (necessarily) extensions of the ground model and they don't (necessarily) have the same ordinals as the ground model. In the realizability model $\mathcal{N}_\varepsilon$, the ordinals form $\simeq$-equivalence classes. In order to realize AC, we should define a choice function on all these ordinals.

What is the problem?



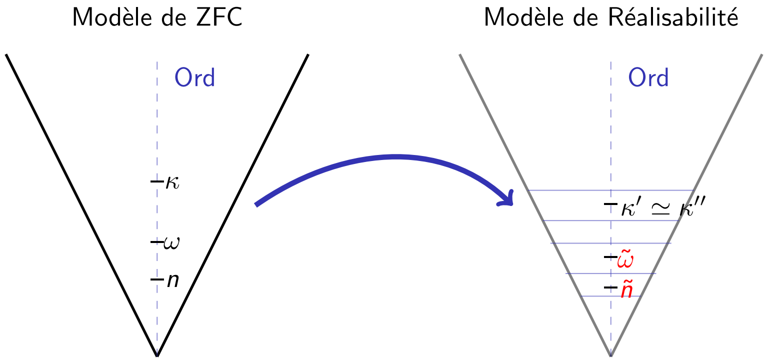
We easily have NEAC, so for AC it would be enough to chose representatives of each ordinal equivalence class, then we could apply NEAC to define a choice function over the representatives and artificially assign the same value to the other ordinals as their representatives.

What is the problem?

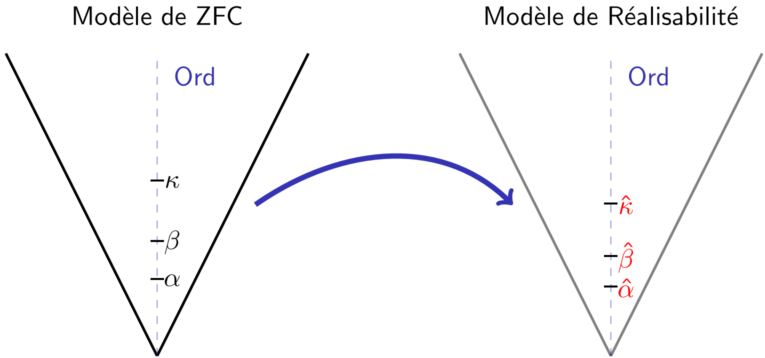


We can easily "represent" the ordinals by using Church numerals (λ-terms) and the instruction 'quote' (Krivine RA2). So we can realize $ZL_\omega = DC$.

What is the problem?



For transfinite ordinals the situation is more delicate (we don't have λ-termes for transfinite ordinals).



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Starting from a model $\mathcal{M}$ of ZF + global choice we can define for every cardinal κ of $\mathcal{M}$ a realizability model where κ has a representative $\hat{\kappa}$ such that $ZF + ZL_{\hat{\kappa}}$ is realized

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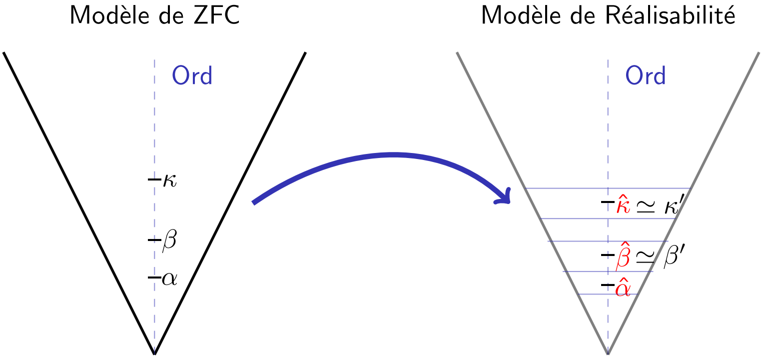

Sketch - the representatives

- We consider an algebra with κ many λ-terms
- We add an instruction χ which “compares the λ-terms by their ordinal index”
- we define for every ordinal $\alpha \leq \kappa$ in the ground model, a set $\hat{\alpha}$
- We show that "hat ordinals" "represent" their counterpart in the ground model.

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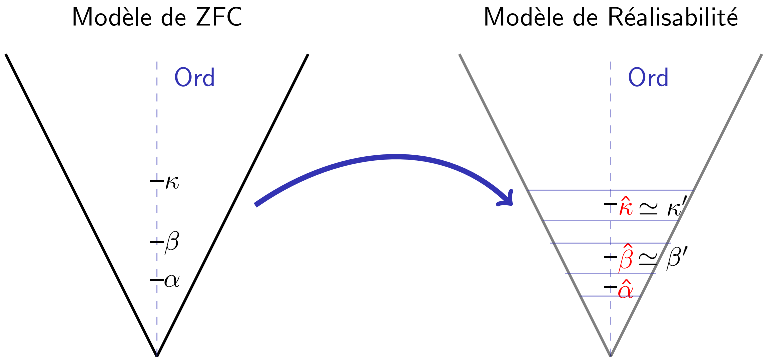

Sketch - realizing $ZL_{\hat{\kappa}}$

- $\hat{\kappa}$ has a $=$-unique representative of each of its $\simeq$-classes of elements
- NEAC entails a choice function over the representatives
- we assign the same value to the other elements in the same class
- we realize $ZL_{\hat{\kappa}}$

Realizing the full Axiom of Choice

Théorème (Krivine, work in progress)

There is a realizability model for the Axiom of Choice (and more)

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but...

... one can show that there is a realizer for AC, but we don't know who is it.

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... one can show that there is a realizer for AC, but we don't know who is it.

Moroever, a theorem of Toschimichi Usuba implies that Krivine's model $\mathcal{N}_\in$ for AC is actually a "small extension" of a model of ZFC: there is a transitive model W of ZFC and a set X such that $\mathcal{N}_\in = V[G]$ for $V = W(X)$, where W is definable in V with parameters in W, and W is a ground of some generic extension of V.

Thank you